Assignment 3

Coverage: 15.4 in Text.

Exercises: 15.4. no 13, 15, 17, 19, 22, 24, 25, 27, 29, 30, 34, 35, 40, 42, 43, 46.

Submit no. 22, 24, 27, 35 by Sept 28.

Supplementary Problems

1. Evaluate the improper integral
$$\int_0^\infty e^{-x^2}\,dx\ .$$

2. Discuss the existence of the improper integral
$$\iint_D \frac{y}{(x^2+y^2)^{3/2}}\ ,$$
where D is the region enclosed by the polar graph $r = 1+\cos\theta$.

Some Definitions and Theorems

Here we re-state some definitions and theorems concerning double integral.

We learned in MATH2010 that a parametric curve is a map from $[a,b]$ to $\mathbb{R}^2$, $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, where γ_1 and γ_2 are continuous functions. It is called a C^1-curve if γ_1 and γ_2 are continuously differentiable. It is called regular if $\gamma_1'^2(t) + \gamma_2'^2(t) > 0$ for all $t \in [a,b]$ (that is, it has a tangent $(\gamma_1'(t), \gamma_2'(t))$ everywhere.) It is closed if $\gamma(a) = \gamma(b)$ (that is, it has no endpoints.) It is simple if $\gamma(t) \neq \gamma(s)$ for distinct $s,t \in [a,b)$, (that is, it has no self-intersection.)

A smooth curve is by definition the image of a parametric C^1-curve. A piecewise smooth curve is a curve consists of n many smooth curves. For instance, taking $n = 2$, there are two parametric C^1-curves $\gamma : [a,c] \to \mathbb{R}^2$ and $\eta : [c,b] \to \mathbb{R}^2$ satisfying $\gamma(c) = \eta(c)$. γ and η together form a piecewise smooth curve. It has a tangent everywhere except possibly at $\gamma(c)$.

A region (or domain) in the plane consists of all points bounded by one or several simple, closed, piecewise smooth curves. For example, the square is bounded by a single such curve consisting of four line segments intersecting at right angle. The annulus $\{(x,y) : 1 \leq x^2+y^2 \leq 4\}$ is bounded by two such curves, that is, the circles $x^2+y^2 = 1$ and $x^2+y^2 = 4$. Points in a region can be divided into interior points and boundary points.

All regions, unless stated explicitly, are bounded.

Most common regions are:

- type I $\{(x,y) :\ a \leq x \leq b,\ f_1(x) \leq y \leq f_2(x)\}$, where f_1 and f_2 are continuously differentiable,
- type II $\{(x,y) :\ c \leq y \leq d,\ g_1(y) \leq x \leq g_2(y)\}$, where g_1 and g_2 are continuously differentiable, and
- those can be decomposed into the union of several type I or type II regions.

The Riemann integral of a function f over a rectangle $[a, b] \times [c, d]$ is defined to be the limit of Riemann sums as their norms tend to 0. The limit may or may not exist. When it does, call the function integrable. It is known that all continuous functions are integrable. For a function f defined in a region D, extend it to be a function in $\mathbb{R}^2$ by setting it zero outside D. The Riemann integral of a function f over a region D is defined to be the Riemann integral of the extended function over some rectangle containing D. Note that even if the function is continuous in D, the extended function may not be continuous at the boundary as a jump across the boundary may occur. Nevertheless, for the three types regions described above, the extended function is still integrable.

Fubini's theorem reduces the evaluation of a double integral to two single integrals. The following version is most common.

Fubini's Theorem. Let D be a type I region. For a continuous function f in D,

$$\iint_D f = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y)\, dydx \ .$$

A similar formula holds for type II regions.